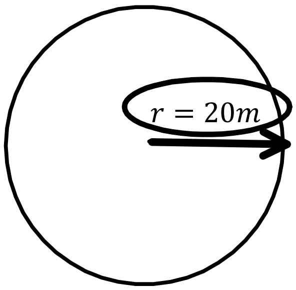
C12 - 4.1 - Related Rates Circle/Sphere A/V Notes

Find the rate of change.

The radius of a circle is growing at a rate of 4 m/s. What is the rate at which the area within the circle is changing when the radius is 20m?



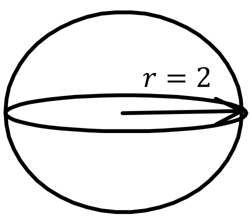
$$\frac{dr}{dt} = 4 \qquad \left.\frac{dA}{dt}\right|_{r=20} = ?$$

$$A = \pi r^2$$
$$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$$
$$\frac{dA}{dt} = 2\pi r \cdot (4)$$
$$\frac{dA}{dt} = 8\pi r$$
$$= 8\pi(20)$$
$$= 160\pi$$
$$\frac{dA}{dt} = 160\pi \frac{m}{s^2}$$

$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt} \qquad \frac{dA}{dr} = 2\pi r \times \frac{dr}{dr}$$

$$\frac{dr}{dr} = 1$$

The volume of a balloon is increasing at 256 meters cubed per second. How fast is the radius increasing when the radius is two meters?



$$\frac{dV}{dt} = 256 \frac{m}{s^3} \qquad \left.\frac{dr}{dt}\right|_{r=2} = ?$$

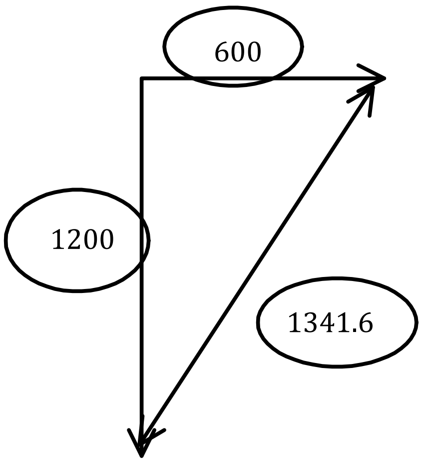
$$V = \frac{4}{3}\pi r^3$$
$$\frac{dV}{dt} = 3 \times \frac{4}{3}\pi r^{3-1} \frac{dr}{dt}$$
$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$
$$256 = 4\pi(2)^2 \frac{dr}{dt}$$
$$\frac{dr}{dt} = \frac{16}{\pi}\frac{m}{s}$$

Therefore the radius is changing at $\frac{16}{\pi}\frac{m}{s}$ when the radius is 2 m.

Therefore the area is changing at a rate of $160\pi \frac{m^2}{s}$ when the radius is $20m$.

C12 - 4.2 - Train Pythag/Spotlight Sim Tri Rel Rat Notes

Train 'a' leaves Vancouver heading South at 10 m/s and train 'b' leaves heading East at 5 m/s? How far are they a part after two minutes? What is the speed at which the trains are moving apart at that time?



$\frac{da}{dt} = 10$ $\quad \frac{db}{dt} = 5$ $\quad \frac{dc}{dt}|_{t=2} = ?$

$$a^2 + b^2 = c^2$$
$$1200^2 + 600^2 = c^2$$
$$c = 1341.6$$

$$a^2 + b^2 = c^2$$
$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt}$$
$$2(1200)(10) + 2(600)(5) = 2(1341.6)\frac{dc}{dt}$$
$$30000 = 2683.2\frac{dc}{dt}$$
$$\frac{dc}{dt} = 11.1\frac{m}{s}$$

$2\ minutes = 120\ seconds$

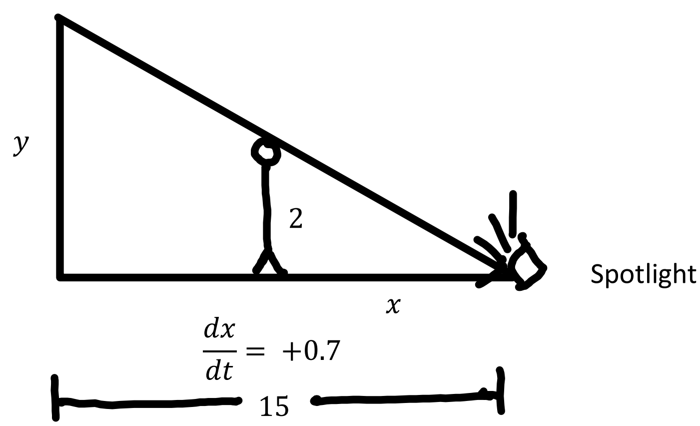
$$a = vt$$
$$a = 10 \times 120$$
$$a = 1200$$

$$b = vt$$
$$b = 5 \times 120$$
$$b = 600$$

$$d = vt$$

A 2 m tall person is walking away from a spotlight, 15 m from a wall, towards the wall at 0.7 m/s. How fast is the shadow on the wall changing when they are 7 m from the spotlight?

$\frac{dy}{dt}|_{x=7} = ?$



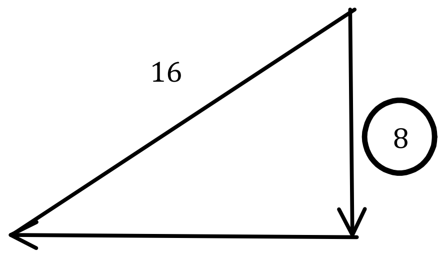
$\frac{dx}{dt} = +0.7$

$$\frac{y}{15} = \frac{2}{x}$$
$$xy = 30$$
$$\frac{dx}{dt}y + \frac{dy}{dt}x = 0$$
$$0.7(4.29) + \frac{dy}{dt}(7) = 0$$
$$\frac{dy}{dt} = -\frac{0.7(4.29)}{7}$$
$$\frac{dy}{dt} = -0.429\frac{m}{s}$$

$$xy = 30$$
$$7y = 30$$
$$y = \frac{30}{7}$$
$$y = 4.29$$

C12 - 4.2 - Ladder Trig Related Rates Notes

The top of a 16 ft ladder slides down a wall at a rate of 3 ft/s. At what rate is the base of the ladder sliding away from the wall when the latter is at a height of 8 ft on the wall.



$$\frac{dy}{dt} = -3\frac{ft}{s}$$

*Length is shrinking: Derivative is Negative.

$$\left.\frac{dx}{dt}\right|_{y=8} = ?$$

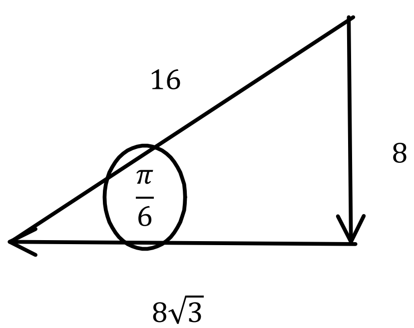
$$x^2 + y^2 = c^2$$
$$x^2 + 8^2 = 16^2$$
$$x = \sqrt{16^2 - 8^2}$$
$$x = \sqrt{192}$$
$$x = 8\sqrt{3}$$

$$x^2 + y^2 = c^2$$
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2c\frac{dc}{dt}$$

*We can substitute constants into the formula

$$2(8\sqrt{3})\frac{dx}{dt} + 2(8)(-3) = 0$$
$$\frac{dx}{dt} = \frac{3}{\sqrt{3}}$$
$$\frac{dx}{dt} = \sqrt{3}\frac{ft}{s}$$

What is the rate the angle at the bottom of the ladder changing?



$$sin\theta = \frac{8}{16}$$
$$\theta = \sin^{-1}(\frac{1}{2})$$
$$\theta = \frac{\pi}{6}$$

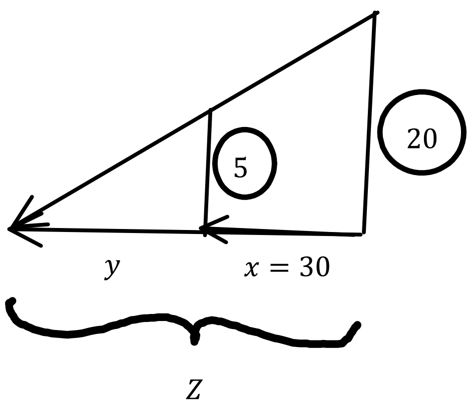
*Real life is in Radians. Degrees are for children.

$$cos\theta = \frac{x}{r}$$
$$cos\theta = \frac{x}{16}$$
$$-sin\theta\frac{d\theta}{dt} = \frac{1}{16}\frac{dx}{dt}$$
$$-\frac{8}{16}\frac{d\theta}{dt} = \frac{1}{16}\sqrt{3}$$
$$\frac{d\theta}{dt} = -\frac{\sqrt{3}}{8}\frac{rad}{s}$$

*I used cos because it used the rate I already solved on the top. Using sin and tan is possible but much more difficult based on the information and previously solved. We want our constant on the bottom.

C12 - 4.2 - Similar Triangles/Cos Law Related Rates Notes

A 5 foot tall woman is walking away from a 20 foot lamp post at 3 m/s. What rate is her shadow increasing when she is 30 feet from the lamp post; and is her shadow getting bigger or smaller. How fast is the tip of her shadow moving?



$$\frac{dx}{dt} = 3\frac{m}{s} \qquad \frac{dy}{dt}\Big|_{x=30} = ?$$

$$\frac{5}{20} = \frac{y}{x+y}$$

$$5x + 5y = 20y$$

$$5x = 15y$$

$$x = 3y$$

$$\frac{dx}{dt} = 3\frac{dy}{dt}$$

$$3 = 3\frac{dy}{dt}$$

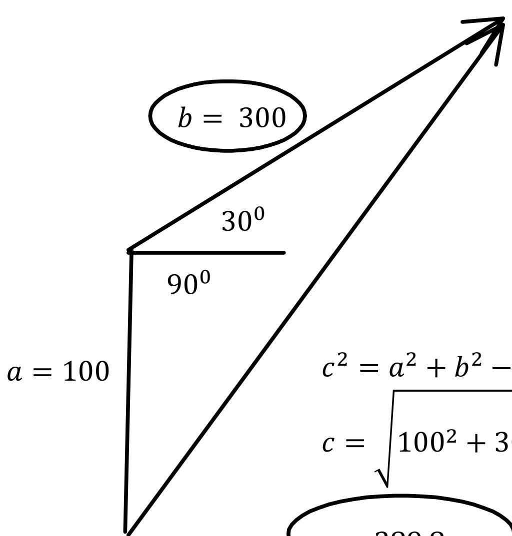
$$\frac{dy}{dt} = 1\frac{ft}{s}$$

$$\frac{dz}{dt} = \frac{dy}{dt} + \frac{dx}{dt}$$

$$\frac{dz}{dt} = 1 + 3$$

$$\frac{dz}{dt} = 4\frac{m}{s}$$

A float plane rising at 30 degrees above the horizontal flies over a boat at an altitude of 100 m at 60 m/s. How fast is the distance between the boat and the plane increasing after five seconds?



$$\frac{db}{dt} = 60 \qquad \frac{dc}{dt}\Big|_{t=5} = ?$$

$$v = \frac{d}{t}$$

$$d = vt$$

$$d = 60 \times 5$$

$$d = 300\ m$$

$$c^2 = a^2 + b^2 - 2abcosC$$

$$c = \sqrt{100^2 + 300^2 - 2(100)(300)\cos\frac{7\pi}{6}}$$

$$c = 389.8\ m$$

*Word Problems in Radians $120^0 = \frac{7\pi}{6}$

*That would have been a tough product rule if more things were changing

$$c^2 = a^2 + b^2 - 2abcosC$$

$$2c\frac{dc}{dt} = 0 + 2b\frac{db}{dt} - 2acosC\frac{db}{dt}$$

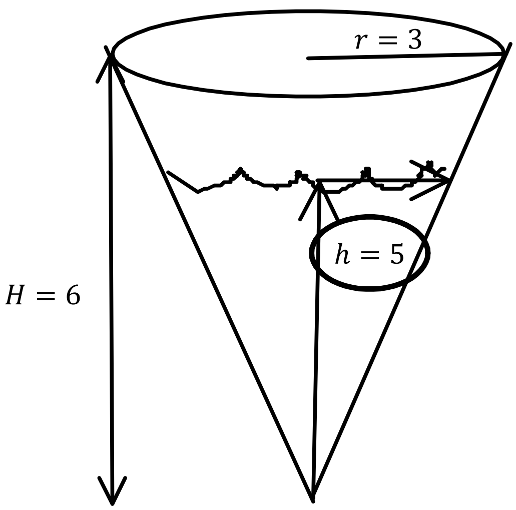
$$2(389.8)\frac{dc}{dt} = 0 + 2(300)(60) - 2(100)\left(-\frac{\sqrt{3}}{2}\right)(60)$$

$$\frac{dc}{dt} = 59.5\frac{m}{s}$$

C12 - 4.3 - Cone V/Similar Triangles Related Rates Notes

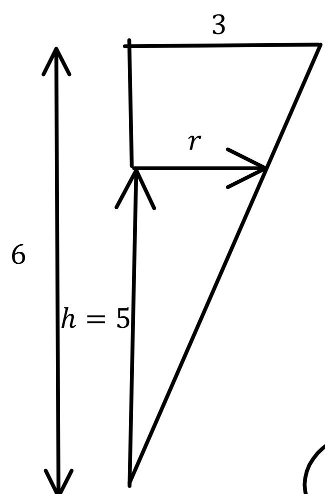
Find the rate of change.

A cone with a radius of 3 cm and height of 6 cm is filling with water where the height of the water level is increasing at a rate of 0.2 cm/s. What is the rate the volume is increasing when the height of the water level is 5 cm.



$$\frac{dh}{dt} = 0.2$$

$$\frac{dV}{dt}\Big|_{h=5} = ?$$



$$\frac{H}{R} = \frac{h}{r}$$

$$\frac{6}{3} = \frac{h}{r}$$

$$2 = \frac{h}{r}$$

$$r = \frac{h}{2}$$

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$$

$$V = \frac{1}{12}\pi h^3$$

$$\frac{dV}{dt} = 3 \times \frac{1}{12}\pi h^2 \frac{dh}{dt}$$

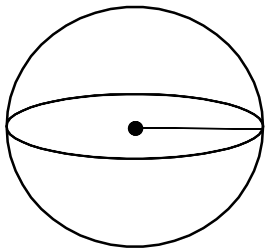
$$\frac{dV}{dt} = \frac{1}{4}\pi(5)^2(0.2)$$

$$\frac{dV}{dt} = \frac{5\pi}{4}\frac{cm}{s}$$

*We can't take this product so we must use similar triangles/other info

$$\frac{dV}{dt} = \frac{1}{3}\pi\left(2r\frac{dr}{dt}h + \frac{dh}{dt}r^2\right)$$

C12 - 4.4 - Sphere Tight Rope Notes

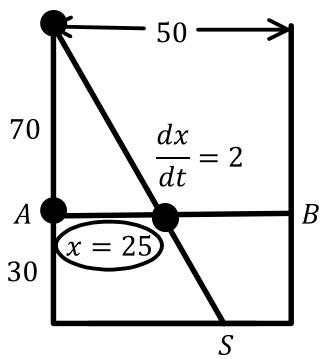


$$\frac{dV}{dt} = ?$$

$$\frac{dr}{dt}\Big|_{SA=20} = 2$$

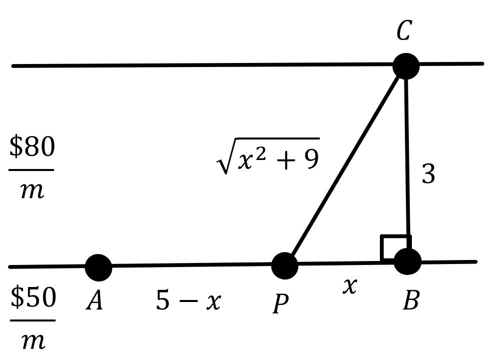
$$V = \frac{4}{3}\pi r^3$$
$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$
$$\frac{dV}{dt} = 4\pi\left(\sqrt{\frac{100}{4\pi}}\right)^2 \frac{dr}{dt}$$
$$\frac{dV}{dt} = 4\pi \times \frac{100}{4\pi} \times 2$$
$$\frac{dV}{dt} = 200\frac{m^3}{s}$$

$$SA = 4\pi r^2$$
$$100 = 4\pi(2)^2$$
$$r = \sqrt{\frac{100}{4\pi}}$$
$$r = \frac{10}{2\sqrt{\pi}}\ m$$



$$\frac{dS}{dt}\Big|_{x=25} = ?$$

$$\frac{x}{70} = \frac{S}{100}$$
$$x = \frac{7}{10}S$$
$$\frac{dx}{dt} = \frac{7}{10}\frac{dS}{dt}$$
$$2 = \frac{7}{10}\frac{dS}{dt}$$
$$\frac{dS}{dt} = \frac{20}{7}\frac{m}{s}$$



$$C = 50(5 - x) + 80\sqrt{x^2 + 9}$$
$$C' =$$
$$0 =$$

Number Line

$$Cost = length \times \frac{cost}{length}$$

C12 - 4.5 - Demand Profit Max

$16\$$ *units sell* 20 *units.*
$14\$$ *units sell* 30 *units.*

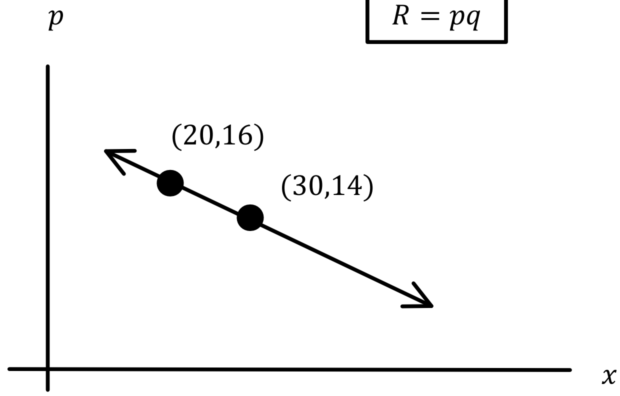
Find q to max R

$$R = pq$$

$$C = 4x + 140$$

$$P = R - C$$

$p = price$
$x = quantity$
$R = Revenue$
$C = Cost$
$P = Profit$



x	p	R	C	P
20	16	*320*	*240*	*120*
30	14	*420*	*260*	*160*

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$m = \frac{16 - 15}{20 - 30}$$
$$m = -\frac{1}{5}$$

Down $1 Sell 5 more

$$y - y_1 = m(x - x_1)$$
$$p - 16 = -\frac{1}{5}(x - 20)$$
$$p = -\frac{1}{5}x + 20$$

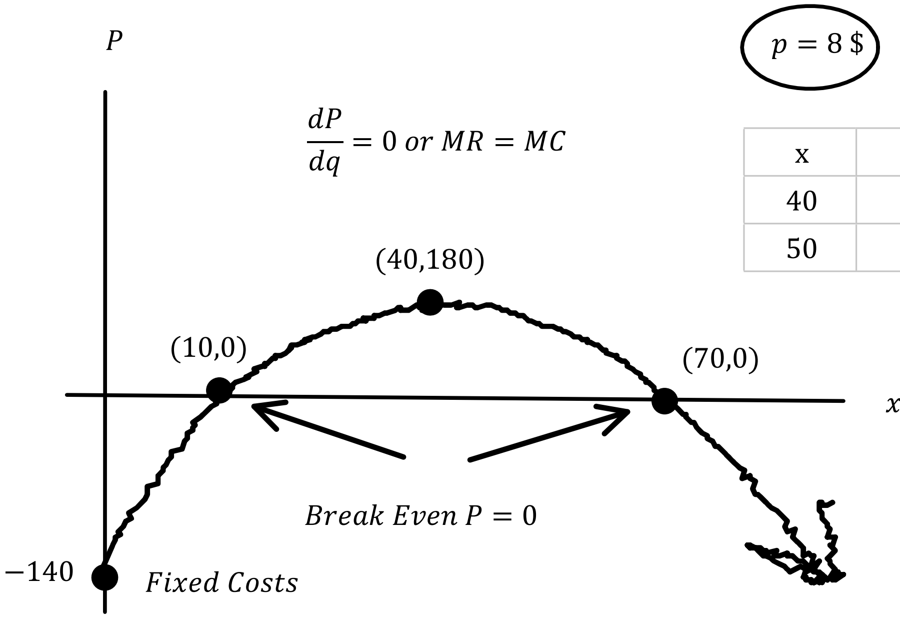
Demand Function

$$R = px$$
$$R = \left(-\frac{1}{5x} + 20\right)x$$
$$R = -\frac{1}{5}x^2 + 20x$$

$$p = -\frac{1}{5}x + 20$$
$$p = -\frac{1}{5} + 20$$
$$p = 8\ \$$$

$$P = R - C$$
$$P = -\frac{1}{5}x^2 + 20x - (4x + 140)$$
$$P = -\frac{1}{5}x^2 + 16x - 140$$
$$\frac{dP}{dx} = -\frac{2}{5}x + 16$$
$$0 = -\frac{2}{5}x + 16$$
$$x = 40\ units$$

$$\frac{dP}{dq} = 0\ or\ MR = MC$$



x	p	R	C	P
40	8	*480*	*300*	*180*
50	10	*500*	*340*	*160*

Max Profit

$$MR = MC$$
$$-\frac{2}{5}x + 20 = 4$$
$$x = 40$$

$$R = pq$$
$$R = (16 - 1x)(20 + 5x)$$
$$R = -5x^2 + 60x + 320$$
$$\frac{dR}{dx} = -10x + 60$$
$$0 = -10x + 60$$
$$x = 6$$

$x = \#\ p\ decreases$ $\quad$ $-\$1\ q\ down, q\ +\ 5$

$$R = -5x^2 + 60x + 320$$
$$R = -5(6)^2 + 60(6) + 320$$
$$R = 500$$

6 price decrease, Max Revenue

Down $\$6, Rev = 500$

C12 - 4.5 - Growth Elasticity Max Rev Notes

$F'(500\$) = ?$; $F(500\$)$ $k = 5\%$ $F = Pe^{kt}$

$$F = Pe^{kt}$$
$$F = Pe^{0.05t}$$
$$F' = Pe^{0.05t} \times 0.05$$

$$F = Pe^{kt}$$
$$F = Pe^{0.05t}$$
$$500 = Pe^{0.05t}$$
$$\frac{500}{P} = e^{0.05t}$$
$$\ln\left(\frac{500}{P}\right) = 0.05tlne$$
$$t = \frac{\ln\left(\frac{500}{P}\right)}{0.05}$$

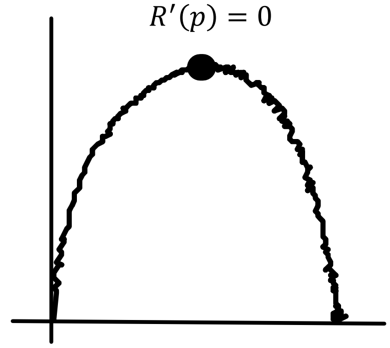
$$F' = Pe^{0.05t} \times 0.05$$
$$F' = Pe^{0.05\frac{\ln\left(\frac{500}{P}\right)}{0.05}} \times 0.05$$
$$F' = Pe^{\ln\left(\frac{500}{P}\right)} \times 0.05$$
$$F' = P\left(\frac{500}{P}\right) \times 0.05$$
$$F' = 25\frac{\$}{year}$$

$$e^{\ln\left(\frac{500}{P}\right)} = \frac{500}{P}$$
$$e^{\ln_e a} = a$$

$q(p) = q$ *Quantity is a function of Price*

$$R = pq$$
$$\frac{DR}{dp} = \frac{dp}{dp}q + \frac{dq}{dp}p$$
Product

Rearrange
$$\frac{DR}{dp} = p\frac{dq}{dp} + q$$
$$\frac{dp}{dp} = 1$$
$$\frac{DR}{dp} = q\left(\frac{p}{q}\frac{dq}{dp} + 1\right)$$
Factor
$$\frac{DR}{dp} = q(E + 1)$$
$$E(p) = \frac{p}{q}\frac{dq}{dp}$$
Elasticity



Price vs Quantity $q + 50p^2 = 240$ Find q to max Rev

$$q + 5p^2 = 240$$
$$\frac{dq}{dp} + 10p\frac{dp}{dp} = 0$$
$$\frac{dp}{dp} = 1$$
$$\frac{dq}{dp} = -10p$$
Sell 10 less each increase in \$p

$$E(p) = \frac{p}{q}\frac{dq}{dp}$$
$$E(p) = \frac{p}{q} \times -10p$$
$$E(p) = -\frac{10p^2}{q}$$
$$-1 = -\frac{10p^2}{q}$$
$$q = 10p^2$$

$E(p) = -1$; @ max

$$q + 50p^2 = 240$$
$$10p^2 + 50p^2 = 240$$
$$60p^2 = 240$$
$$p^2 = 4$$
$$p = 2$$

$$q = 5p^2$$
$$q = 5(2)^2$$
$$q = 20$$